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HEAT CONDUCTION IN SOLIDS: TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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Abstract—A variational formulation is derived and applied to problems in heat conduction through solids. This formulation has, as its Euler–Lagrange equation, the general heat equation where the thermal conductivity, density, and specific heat may be temperature-dependent. The variational principle can be applied for steady-state or time-dependent solutions and is valid for boundary conditions where the temperature is specified, being either fixed or time-dependent, or where the heat flux is zero. The macroscopic temperature distribution for two problems is determined using this technique. The first of these is a finite rectangular plate, the initial temperature of the plate being constant and the temperature of the edges being set equal to zero for time greater than zero. In the second example the variational formulation is used to obtain a steady-state temperature distribution for a finite rectangular plate with sides held at fixed but different temperatures. In both of these applications, the thermal conductivity is assumed to vary linearly with temperature and the density and specific heat are assumed to have uniform values. Comparison of the results obtained using the variational formulation with those obtained by finite difference techniques shows an excellent correlation over a range of the temperature-conductivity coefficient.

NOMENCLATURE

K, thermal conductivity;
R, rectangle length–width ratio;
S, boundary surface;
T, temperature;
V, volume;
a, b, length of rectangle sides;
c_v, specific heat;
t, time;
x_i, coordinate axes.

Greek symbols
θ, dimensionless temperature;
α_n, β_n, trial function coefficients;
δ, variational symbol;
δ_{ij}, Kronecker delta;
κ, thermal diffusivity;
ρ, density;
σ, slope of dimensionless temperature—diffusivity curve;
τ, dimensionless time;
ξ, ϕ, dimensionless coordinate axes.

Subscript
0, reference value.

Superscript
$*$, variables not subject to variation.

1. INTRODUCTION

THE USEFULNESS of the concept of entropy production as a measure of the approach to equilibrium has been recognized for many years and has been discussed in detail by Prigogine [1]. Over a period of years there has evolved a relatively new concept which was termed the "local potential" by Glansdorff, Prigogine and Hays [2] and which results in a variational principle which was applicable to systems exhibiting reversible as well as irreversible processes. This work was extended by Glansdorff and Prigogine [3] to include a very general evolution criterion for processes in macroscopic physics with time-independent boundary conditions. A physical interpretation of the local potential concept was obtained by Prigogine [4] and Glansdorff [5] where the existence and function of the concept was explained in the light of fluctuation theory. It was then possible to explain on a rational basis the unique characteristic of this formulation; the inclusion of functions in the Lagrangian which are evaluated at the steady state and thus not subject to variation. Although the variational formulation had been derived on the basis of time-independent boundary conditions, it was shown by Hays [6] that this restriction was not necessary and that the formulation was valid for time-dependent as well as time-independent boundary conditions. These extended variational principles have been applied successfully by Hays to several problems in hydrodynamics [7, 8], to the flow of heat in solids [6] and to diffusion [9]. Since the work on heat flow did not include two-dimensional temperature distributions it seemed desirable to evaluate the applicability of the variational formulation for these more involved applications. This investigation is for the purpose of acquiring a greater understanding of the applicability of the variational method as applied to two types of problems in heat conduction in solids and also to evaluate its usefulness with respect to other techniques in achieving solutions to these problems.

The variational formulation that is applicable to the general diffusion or heat-conduction equation, as derived by the author [6], is based upon the following concepts. Consider a volume V of an isotropic solid with a boundary surface S. The family of temperature distributions in this volume are thought of as having the appropriate macroscopic temperature distribution T^* plus small and arbitrary variations of the temperature δT around the macroscopic distribution, both components being functions of space and time,

$$T = T^*(x_i, t) + \delta T(x_i, t). \tag{1.1}$$

The thermal conductivity K, density ρ, and specific heat c_v are assumed to be functions of the temperature T and consequently can be written as

$$\begin{aligned} K(T) &= K(T^* + \delta T) = K^* + \delta K, \\ \rho &= \rho^* + \delta\rho, \\ c_v &= c_v^* + \delta c_v. \end{aligned} \tag{1.2}$$

If on the boundary surface S the temperature is specified ($\delta T = 0$) or if the flux across the surface is zero, then it can be shown that the following variation is always less than zero and is equal to zero when the temperature distribution corresponding to T^* is reached;

$$\int_t\int_V \left[\frac{\partial T^*}{\partial t}\,\delta T + K^*\frac{\partial}{\partial x_i}\left(\frac{\delta T}{\rho^* c_v^*}\right)\frac{\partial T}{\partial x_i}\right] \mathrm{d}V\,\mathrm{d}t \leqslant 0. \tag{1.3}$$

The required macroscopic temperature distribution T^* is characterized by the extremum condition

$$\left[\frac{\delta \mathscr{L}(T^*, T)}{\delta T}\right]_{T^*} = 0 \tag{1.4}$$

with the subsidiary condition

$$T = T^*. \tag{1.5}$$

If the density and specific heat are independent of temperature, as in the examples to be considered in this paper, then (1.3) can be simplified. A functional J is defined

$$J = \int_t\int_V \left[\frac{K^*}{2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \rho^* c_v^*\, T\left(\frac{\partial T^*}{\partial t}\right)\right] \mathrm{d}V\,\mathrm{d}t, \tag{1.6}$$

which has as its Euler–Lagrange equation the equation for the conservation of energy where the thermal conductivity is temperature-dependent and the density and the specific heat are constant. This relationship can be easily shown by setting the first variation of (1.6) equal to zero.

The variational formulation (1.6) is applicable to both steady-state and time-dependent heat-conduction problems with either fixed or time-dependent boundary conditions. The only restriction on the above formulation is that either the temperature be specified on the boundary surface S, i.e. $\delta T = 0$, or that the flux across this surface be zero.

The integral (1.6) is a function of both the macroscopic temperature distribution T^* and the total temperature T and it is only the latter distribution which is subject to variation. The phenomenological coefficients ρ^*, c_v^*, and K^* appearing in the formulation may be arbitrary functions of temperature which are evaluated with respect to the temperature T^*. Consequently they are not subject to variation.

A fundamental problem in the application of variational techniques is the choice of an appropriate trial function. Such a trial function must be capable of representing, or approximating, the actual solution of the differential equation and it must not be so complex as to cause undue mathematical hardship. It is assumed that the trial function will contain a number of undetermined coefficients $\alpha_{n,m}$, $n = 1, 2, \ldots, N$; $m = 1, 2, \ldots, N$, and it will satisfy the boundary conditions of the system being investigated. In this paper, the "self-consistent" or "direct" method is applied to obtain the $\alpha_{n,m}$. The function T^* is assumed to consist of undetermined coefficients $\beta_{n,m}$ while the function T consists of the coefficients $\alpha_{n,m}$, the general form of the space–time representation of both functions being the same. Using the Rayleigh–Ritz method, a series of N^2 algebraic equations is formed as a consequence of the minimization process,

$$\frac{\partial J}{\partial \alpha_{i,j}} = 0, \qquad i,j = 1, 2, \ldots, N. \tag{1.7}$$

The subsidiary condition, $T = T^*$, is now effected by setting $\alpha_{n,m} = \beta_{n,m}$ and the N^2 algebraic equations are then solved for the coefficients $\alpha_{n,m}$. In general, recourse must be made to numerical methods to determine the coefficients.

2. TIME-DEPENDENT TEMPERATURE DISTRIBUTIONS

The first application of the variational principle is to the problem of thermal conduction in a solid isotropic material, $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. This rectangular plate is of uniform temperature T_s for $t < 0$ and the temperature of the edges, $x = 0$, $x = a$, $y = 0$, and $y = b$, is maintained at zero for $t > 0$. In this problem the density ρ of the material and the specific heat c_v are considered to be uniform. These terms are combined with the temperature-dependent thermal conductivity, thus defining a temperature-dependent thermal diffusivity κ.

$$\kappa = \frac{K(T)}{\rho c_v}. \tag{2.1}$$

This example will therefore test the applicability of the variational formulation to problems where the solutions are time-dependent with two space variables, $T(x, y; t)$ and where the boundary conditions are specified.

It is advantageous to change variables in the following manner.

$$\xi = \frac{\pi x}{a}, \qquad \phi = \frac{\pi y}{b}, \qquad \tau = \frac{\kappa_0 t}{a^2}, \qquad \theta = \frac{T}{T_s}, \qquad R = \frac{a}{b}. \tag{2.2}$$

The conservation of energy equation then takes the following form

$$\frac{\partial \theta}{\partial \tau} = \pi^2 \left\{ \frac{\partial}{\partial \xi}\left(\frac{\kappa}{\kappa_0}\frac{\partial \theta}{\partial \xi}\right) + R^2 \frac{\partial}{\partial \phi}\left(\frac{\kappa}{\kappa_0}\frac{\partial \theta}{\partial \phi}\right)\right\} \tag{2.3}$$

and the initial and boundary conditions are as follows.

$$\left.\begin{array}{l} 0 \leqslant \xi \leqslant \pi, \quad 0 \leqslant \phi \leqslant \pi, \quad \tau < 0; \quad \theta = 1 \\ \xi = 0, \pi; \quad \phi = 0, \pi; \quad \tau > 0; \quad \theta = 0. \end{array}\right\} \tag{2.4}$$

The thermal diffusivity κ is assumed to be a linear function of temperature where σ is the slope of the dimensionless temperature–diffusivity curve and κ_0 is a reference value.

$$\kappa = \kappa_0(1 + \sigma\theta). \tag{2.5}$$

The variational form (1.6) in terms of the new variables becomes†

$$J = \int_0^\infty \int_0^\pi \int_0^\pi \left\{ \frac{\pi^2(1+\sigma\theta^*)}{2} \left[\left(\frac{\partial\theta}{\partial\xi}\right)^2 + R^2 \left(\frac{\partial\theta}{\partial\phi}\right)^2 \right] + \theta \frac{\partial\theta^*}{\partial\tau} \right\} d\xi \, d\phi \, d\tau. \qquad (2.6)$$

This form is applicable since the boundary temperatures are specified and the surface integral vanishes. It is now necessary to assume a trial function for the temperature θ. In making this choice it is desirable to obtain a relatively simple form which will have the ability to adequately represent the desired solutions. The variational process is one of approximation and the final results will be the best fit of the trial function to the solution of the differential equation. The trial function is taken to be

$$\theta = \frac{16}{\pi^2} \sum_m^\infty \sum_n^\infty \frac{1}{mn} \sin n\xi \sin m\phi \exp(-\alpha_{n,m}\tau) \quad m, n, = 1, 3, 5, \ldots, \infty \qquad (2.7)$$

which satisfies the boundary conditions (2.4). The coefficients appearing in the exponential term must be determined such that the integral (2.6) is minimized. Consequently, the partial derivative of the integral J (2.6) is taken with respect to the specific coefficients $\alpha_{k,l}$ and the resulting time–space integral is set equal to zero. In taking this derivative the functions with the superscript * are not subject to variation.

$$\frac{\partial J}{\partial \alpha_{kl}} = \int_0^\infty \int_0^\pi \int_0^\pi \left\{ \pi^2(1+\sigma\theta^*) \left[\left(\frac{\partial\theta}{\partial\xi}\right) \frac{\partial}{\partial\alpha_{kl}} \left(\frac{\partial\theta}{\partial\xi}\right) + R^2 \left(\frac{\partial\theta}{\partial\phi}\right) \frac{\partial}{\partial\alpha_{kl}} \left(\frac{\partial\theta}{\partial\phi}\right) \right] + \left(\frac{\partial\theta}{\partial\alpha_{kl}}\right) \left(\frac{\partial\theta^*}{\partial\tau}\right) \right\} d\xi \, d\phi \, d\tau = 0. \qquad (2.8)$$

The subsidiary condition (1.5) is now applied by setting $\theta = \theta^*$ and the trial function (2.7) is used to determine the several terms appearing in (2.8). The square of an infinite series is required in evaluating (2.8) and, to avoid confusion in the mathematical operations, the subscripts s and t are assigned to one series, s, t, n, and m covering the same range. The resulting set of algebraic equations is in terms of the coefficients $\alpha_{k,l}$.

$$\frac{\pi^4}{16}\left(\frac{1}{l^2} + \frac{R^2}{k^2}\right) - \frac{\pi^2 \alpha_{kl}}{16k^2l^2} - 128\sigma \sum_n \sum_m \sum_s \sum_t \frac{(s^2 - k^2 - n^2)}{[s^2-(k+n)^2][s^2-(k-n)^2][t^2-(m-l)^2][t^2-(m+l)^2]} \times \left(\frac{\alpha_{kl}}{\alpha_{st}+\alpha_{nm}+\alpha_{kl}}\right)^2 - 128\sigma R^2 \sum_n \sum_m \sum_s \sum_t \frac{(t^2 - m^2 - l^2)}{[s^2-(k-n)^2][s^2-(k+n)^2][t^2-(m+l)^2][t^2-(m-l)^2]} \times \left(\frac{\alpha_{kl}}{\alpha_{st}+\alpha_{nm}+\alpha_{kl}}\right)^2 = 0 \quad n, m, s, t, = 1, 3, 5, \ldots, \infty. \qquad (2.9)$$

† In this example the limits on the time integral are taken to be zero and infinity. Thus the fit of the trial function will be over all time. A referee has noted, however, that since the coefficients in the trial function are determined by the values of the limits of integration, the fit may be made over contiguous intervals. This poses the problem of the function being discontinuous at the interval boundaries.

These equations (2.9), truncated at the Nth term in m and n, are of the form

$$F_{ij}(\alpha_{11}, \alpha_{12}, \ldots \alpha_{1,N}; \alpha_{21}, \alpha_{22}, \ldots \alpha_{2,N}; \ldots \alpha_{N1}, \alpha_{N2}, \ldots \alpha_{N,N}) = 0 \qquad i, j = 1, 2, 3, \ldots, N. \qquad (2.10)$$

The Newton–Raphson method (Booth [10]) was used to determine these coefficients. Since this technique was discussed in detail in [6], it will not be repeated here.

In general, it is not possible to assume arbitrary initial values for the α's and be assured that the Newton–Raphson method will converge to the correct solution. The trial values must be in the vicinity of those corresponding to the solution. Some preliminary exploration as to where the roots are located is advisable before attempting to obtain a solution. By considering the case of uniform thermal conductivity, $\sigma = 0$, it is possible to obtain an initial set of $\alpha_{n,m}$ values. Using these starting values it is then possible to obtain α's for small σ values which in turn can be used as starting values for yet larger values of the conductivity coefficient.

When $\sigma = 0$ in (2.9), then the α_{kl} are easily obtained,

$$\alpha_{kl} = \pi^2(k^2 + R^2 l^2), \qquad k, l = 1, 3, 5, \ldots, N. \qquad (2.11)$$

The substitution of (2.11) into (2.7) results in the following temperature function for uniform conductivity.

$$\theta = \frac{16}{\pi^2} \sum_k \sum_l \frac{1}{kl} \sin k\xi \sin l\phi \; \exp\left[-\pi^2(k^2 + R^2 l^2)\tau\right]. \qquad (2.12)$$

This is a well-known solution (Carslaw and Jaeger [11]). The α's given by (2.11) were used as initial values for $\sigma \ll 1$. Then, with σ incremented in small steps, the α's resulting from one solution were used as the starting values for the next iteration, until all values of σ which were of interest had been obtained. The iterations were terminated when the differences between all α's for two successive iterations were less than 1 per cent. When values of α_{kl} corresponding to $\sigma = 0$ and $\sigma = 0{\cdot}5$ and a value of $R = 1$ (square plate) are substituted into (2.7), temperature distributions are obtained as shown in Fig. 1 for

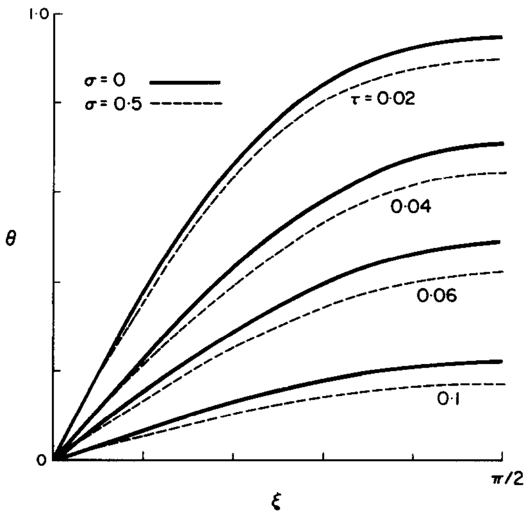

FIG. 1. Temperature distributions θ across the semi-width of a finite rectangular plate at the mid-section $\phi = \pi/2$ for a uniform thermal conductivity $\sigma = 0$ and a variable conductivity $\sigma = 0{\cdot}5$ for four time values τ; $R = 1$.

several values of the dimensionless time τ. These temperature distributions are across the semi-width of the plate at the mid-section $\phi = \pi/2$. It is seen that appreciable differences exist between the solutions where the thermal conductivity variation with temperature is taken into account and where it is considered to be uniform.

In Fig. 2 are shown temperature distributions over the semi-length of the plate at the centerline $\xi = \pi/2$ for the dimensionless time $\tau = 0{\cdot}04$. These curves are for two width-to-length ratios, $R = 1$ and $R = \frac{1}{2}$. The classical method of solution where the conductivity is considered to be uniform would lead to appreciable errors as shown by this comparison with the solution based on a variable thermal diffusivity.

To evaluate the usefulness of the variational formulation in achieving solutions to thermal conduction problems, it is necessary to have a reference solution for comparison. However, there are few solutions available which can be used as a reference because of the difficulty in obtaining solutions to problems with variable

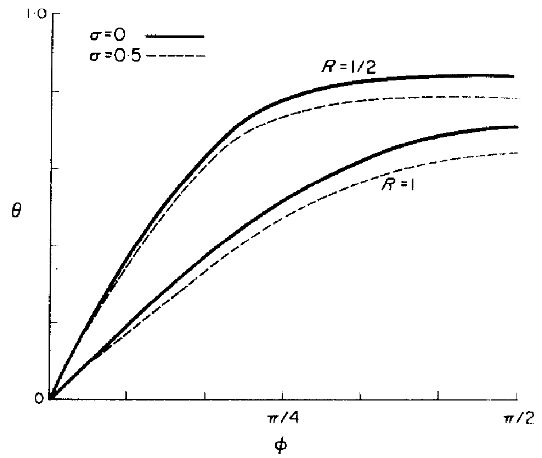

FIG. 2. Temperature distributions θ over the semi-length of a finite rectangular plate along the centerline for a uniform thermal conductivity $\sigma = 0$ and a variable conductivity $\sigma = 0{\cdot}5$ for the dimensionless time $\tau = 0{\cdot}04$ and for two width-to-length ratios, $R = 1$ and $R = \frac{1}{2}$.

thermal conductivities. Consequently, the solution to this problem was obtained using finite difference methods (alternating direction) as described by Douglas [12]. These solutions were obtained on a high-speed digital computer and are considered as reference solutions.

In Fig. 3 are shown comparisons of temperature distributions from the variational method and the finite difference method. In this figure $\sigma = 0{\cdot}5$, and it is seen that the variational solutions compare favorably with the solutions obtained through finite difference techniques. The discrepancies shown between the two techniques are the results of the influence of σ on the trial function which is to be found only in the time factor. A more refined trial function which would have greater flexibility in matching these solutions could be used but at the expense of making the analysis more complex. However, inasmuch as the variational approach seeks to find approximate solutions, one must conclude on the basis of these comparisons that it is extremely successful in fulfilling its purpose.

Instead of using a trial function (2.7) which requires the determination of a multitude of coefficients α_{kl}, we will now investigate the possibility of a trial function requiring but a single coefficient α. In this case a possible trial function would be of the form

$$\theta = \frac{16}{\pi^2} \sum_n \sum_m \frac{1}{nm} \sin n\xi \sin m\phi \exp\left[-\alpha\pi^2(n^2 + R^2m^2)\tau\right] \qquad (2.13)$$

where a single α appears in the exponential term. The minimization is effected as before by performing the integrations indicated in the following expression,

$$\frac{\partial J}{\partial \alpha} = \int_0^\infty \int_0^\pi \int_0^\pi \left\{ \pi^2(1 + \sigma\theta^*) \left[\left(\frac{\partial\theta}{\partial\xi}\right) \frac{\partial}{\partial\alpha}\left(\frac{\partial\theta}{\partial\xi}\right) + R^2 \left(\frac{\partial\theta}{\partial\phi}\right) \frac{\partial}{\partial\alpha}\left(\frac{\partial\theta}{\partial\phi}\right) \right] + \left(\frac{\partial\theta}{\partial\alpha}\right)\left(\frac{\partial\theta^*}{\partial\tau}\right) \right\} d\xi \, d\phi \, d\tau. \qquad (2.14)$$

A single equation is now available for the determination of α.

$$\frac{\pi^2}{16} \sum_n \sum_m \frac{\alpha - 1}{n^2m^2} + \frac{128\sigma}{\pi^2} \sum_n \sum_m \sum_p \sum_q \sum_s \sum_t \frac{(n^2 + R^2m^2)}{[n^2 + p^2 + s^2 + (m^2 + q^2 + t^2)R^2]^2} \times \left\{ \frac{(p^2 - n^2 - s^2)}{[(p+s)^2 - n^2][(p-s)^2 - n^2][q^2 - (t-m)^2][q^2 - (t+m)^2]} \right.$$

$$+ \frac{R^2(q^2 - m^2 - t^2)}{[n^2 - (p - s)^2][n^2 - (p + s)^2][(q + t)^2 - m^2][(q - t)^2 - m^2]}\Bigg\} = 0,$$

$$n, m, p, q, s, t = 1, 3, 5, \ldots, \infty. \qquad (2.15)$$

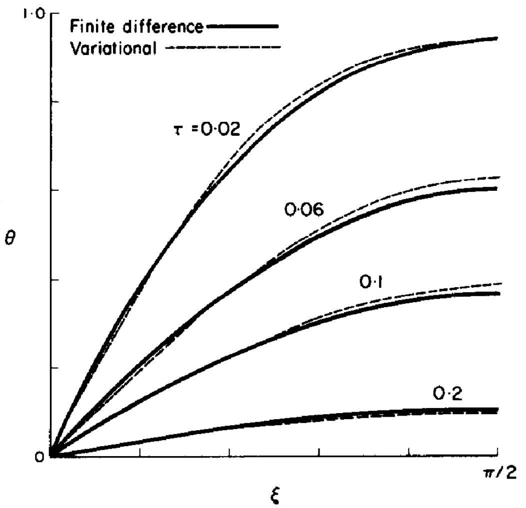

FIG. 3. Temperature distributions θ across the semi-width of a finite rectangular plate at the mid-section $\phi = \pi/2$ for a variable thermal conductivity $\sigma = 0{\cdot}5$ obtained using the variational formulation with twenty-five coefficients, compared to temperature distributions obtained through finite difference methods; $R = \frac{1}{2}$.

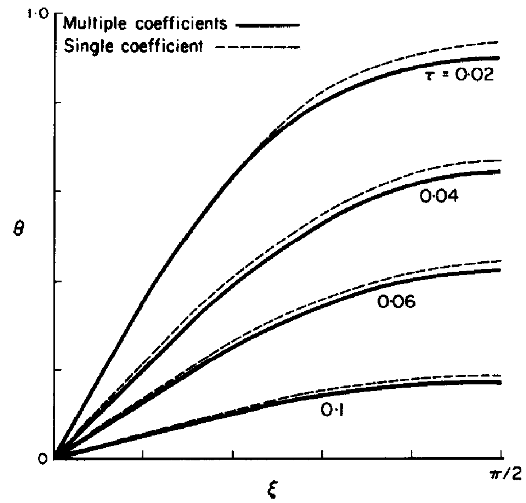

FIG. 4. Temperature distributions θ across the semi-width of a finite rectangular plate at the mid-section $\phi = \pi/2$ for a variable thermal conductivity $\sigma = 0{\cdot}5$ obtained using the variational formulation with twenty-five coefficients, compared with those obtained with a single coefficient; $R = 1$.

The multiple α's and the single α were substituted into (2.7) and (2.13), respectively, and Fig. 4 shows comparisons between the two methods. These temperature distributions are along the semi-width of the plate at the mid-section $\phi = \pi/2$ for a conductivity coefficient $\sigma = 0{\cdot}5$ and a width-to-length ratio $R = 1$. The agreement between the temperature distributions determined by a single α and by twenty-five α's is close over a wide range of the dimensionless time τ. The temperature distributions determined by the single α are slightly higher in all cases than those distributions determined by the multiple α's. The multiple α's were always found to yield a more accurate solution.

3. TIME-INDEPENDENT TEMPERATURE DISTRIBUTION

In the previous example the variational formulation was used to achieve solutions which were time-dependent with two space variables. In this example, that of a finite rectangular plate, the variational principle will be used to obtain a solution of a steady-state problem in two dimensions where the temperatures are fixed but different on the boundaries. The region of interest extends over $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$, the temperature on all of the boundaries being zero with the exception of $T(x, b) = T_s \sin \pi x/a$. The thermal conductivity is again assumed to be linearly dependent on temperature and the density and specific heat are assumed to be uniform. With the following change of variables

$$\xi = \frac{\pi x}{a}, \qquad \phi = \frac{\pi y}{b}, \qquad \theta = \frac{T}{T_s}, \qquad (3.1)$$

the energy equation and boundary conditions are given by

$$\frac{\partial}{\partial \xi}\left(\frac{\kappa}{\kappa_0}\frac{\partial \theta}{\partial \xi}\right) + R^2 \frac{\partial}{\partial \phi}\left(\frac{\kappa}{\kappa_0}\frac{\partial \theta}{\partial \phi}\right) = 0, \qquad (3.2)$$

$$\theta(\xi, \pi) = \sin \xi, \qquad \theta(\xi, 0) = \theta(0, \phi) = \theta(\pi, \phi) = 0. \qquad (3.3)$$

The variational formulation to be used is

$$J = \int_0^\pi \int_0^\pi (1 + \sigma\theta^*) \left[\left(\frac{\partial \theta}{\partial \xi}\right)^2 + R^2 \left(\frac{\partial \theta}{\partial \phi}\right)^2\right] \mathrm{d}\xi\, \mathrm{d}\phi \qquad (3.4)$$

with the thermal diffusivity given by (2.5).

A trial function for θ which satisfies the boundary conditions is

$$\theta = \frac{\phi}{\pi} \sin \xi + \sum_n \sum_m \beta_{nm} \sin n\xi \sin m\phi$$

$$n = 1, 3, 5, \ldots \infty \qquad (3.5)$$

where the β_{nm} are the coefficients to be determined. The derivative of (3.4) is taken with respect to the particular set of coefficients β_{kl} and set equal to zero. The subsidiary condition (1.5) is now satisfied by setting $\theta = \theta^*$.

After the appropriate integrations and algebraic manipulations the following infinite set of algebraic equations is obtained for the β_{kl}.

$$\begin{aligned}
&\frac{-\pi}{2l}(-1)^l + \frac{\pi^2}{4}\beta_{kl}(k^2 + R^2 l^2) + \frac{4R^2\sigma}{\pi^2 kl(k^2 - 4)}[1 - (-1)^l] \\
&+ \frac{\sigma k}{\pi^2(4 - k^2)}\left\{\frac{\pi^2 l^2 - 4}{l^3}[1 - (-1)^l] - \frac{\pi^2}{l}[1 + (-1)^l]\right\} \\
&+ \frac{2\sigma}{\pi}\sum_n \sum_m kn\beta_{nm}\frac{(1 - n^2 - k^2)}{[1 - (n + k)^2][1 - (n - k)^2]}\left\{\frac{\pi^2}{4}\delta_{ml} - \frac{2lm\Omega_{ml}}{(l^2 - m^2)^2}[1 - (-1)^{m+l}]\right\} \\
&- \frac{4R^2\sigma}{\pi}\sum_n \sum_m \beta_{nm}\frac{klmn}{[1 - (n - k)^2][1 - (n + k)^2]}\left\{\frac{\pi^2}{4}\delta_{ml} - \frac{(l^2 + m^2)\Omega_{ml}}{(l^2 - m^2)^2}[1 - (-1)^{m+l}]\right\} \\
&+ \frac{2\sigma}{\pi}\sum_n \sum_m \beta_{nm}\frac{kn(n^2 - k^2 - 1)}{[(n + 1)^2 - k^2][(n - 1)^2 - k^2]}\left\{\frac{\pi^2}{4}\delta_{ml} - \frac{2ml\Omega_{ml}}{(l^2 - m^2)^2}[1 - (-1)^{m+l}]\right\} \\
&- 8\sigma\sum_n \sum_m \sum_s \sum_t \beta_{nm}\beta_{st}\frac{klmnst(s^2 - k^2 - n^2)}{[(s + n)^2 - k^2][(s - n)^2 - k^2][l^2 - (m - t)^2][l^2 - (m + t)^2]} \\
&- \frac{4R^2\sigma}{\pi}\sum_n \sum_m \beta_{nm}\frac{klmn\Omega_{ml}}{[1 - (n - k)^2][1 - (n + k)^2](m^2 - l^2)}[1 - (-1)^{m+l}] \\
&- 8\sigma R^2\sum_n \sum_m \sum_s \sum_t \beta_{nm}\beta_{st}\frac{klmnst(t^2 - l^2 - m^2)}{[k^2 - (n - s)^2][k^2 - (n + s)^2][(t + m)^2 - l^2][(t - m)^2 - l^2]} = 0
\end{aligned} \qquad (3{\cdot}6)$$

where $\Omega_{ij} = (1 - \delta_{ij})$
and δ_{ij} is the Kronecker delta.

An initial set of trial values for β_{nm} is obtained by setting $\sigma = 0$ in (3.6) and solving for the coefficients. Thus

$$\beta_{1,l} + \frac{2(-1)^l}{\pi l(1 + R^2 l^2)}. \qquad (3.7)$$

For the case of uniform conductivity, $\sigma = 0$, there exists but one k value, that of unity. The coefficients β_{kl} from (3.7) are used as initial values to allow one to proceed with the evaluation of the coefficients for the case of variable diffusivity.

In Fig. 5 are shown steady-state temperature distributions for selected values of ϕ, comparisons being made between the solutions with variable conductivity, $\sigma = 0{\cdot}5$, and uniform conductivity $\sigma = 0$. In Fig. 6 comparisons are

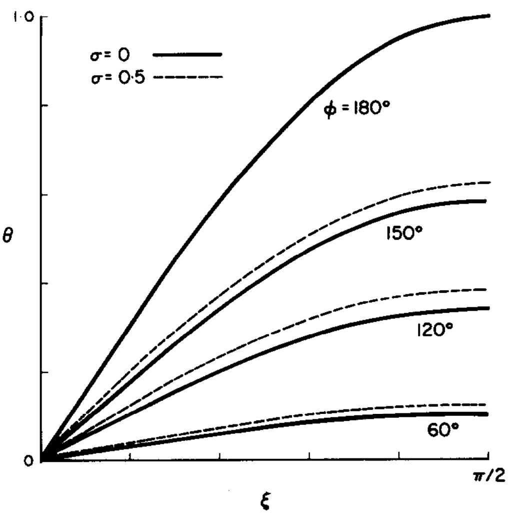

FIG. 5. Steady-state temperature distributions across the semi-width of a finite rectangular plate for selected values of ϕ for a uniform thermal conductivity $\sigma = 0$ and a variable conductivity $\sigma = 0{\cdot}5$, $R = 1$.

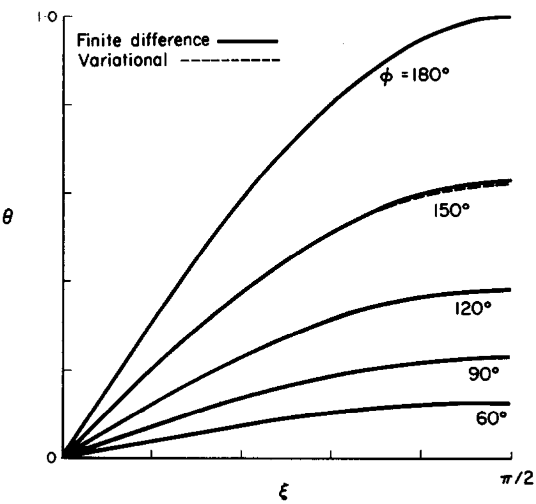

FIG. 6. Steady-state temperature distributions across the semi-width of a finite rectangular plate for a variable thermal conductivity $\sigma = 0{\cdot}5$ obtained using the variational formulation, compared with those obtained through the use of finite difference techniques; $R = 1$.

made between the solutions obtained from the variational formulation and those obtained using finite difference techniques. In this case the overrelaxation method of Young [13] was used to obtain the reference solutions. For these steady-state solutions, excellent correlation is obtained between the variational and the reference solutions over a wide range of ϕ values.

CONCLUSIONS

These two examples have shown that it is possible to use an extended variational formulation to achieve solutions to problems with temperature-dependent diffusivities. In these examples we have also seen that the approximate solutions obtained by the variational method are dependent on the form of the assumed trial function. If knowledge is available relating to the behavior of the solutions, then this knowledge can be incorporated into the trial function. If information is lacking, then there is the possibility of approximating the solutions by other methods which will yield sufficient information as to the characteristics of the solutions to allow a reasonable assumption for the trial function.

The greatest advantage of the variational principle as applied to heat-conduction problems is that its applicability is not limited by the manner in which the thermal conductivity, density, or specific heat are temperature-dependent. The temperature dependence of these phenomenological coefficients, when known, can then be incorporated directly into the variational formulation. If these relationships should be other than the simple linear form used in this analysis, this would result in additional algebraic operations because of the higher order terms, however, the basic mathematical operations would remain unchanged.

The application of the variational technique requires the availability of a computer facility, but this same requirement is found with regard to other methods currently used to achieve solutions to the types of problems that have been used as examples. The amount of computer time required to achieve solutions through either finite difference methods or variational techniques can vary appreciably. It is not possible to give absolute comparisons because of the various trial functions that might be used and the differing finite difference techniques that can be applied. It was found that the computer time required by the variational method in achieving the steady-state solutions for the finite plate was approximately one-half the time required by the finite difference methods. For the time-dependent solutions, however, the variational formulation exceeded, by a factor of one-half, the time required by the finite difference method. If a judicious choice of the trial function is made or if the appropriate finite difference method is chosen, then little difficulty is found in achieving the required temperature distributions. However, a poor choice of either of these can lead one to a great many difficulties in this type of problem.

Although the Newton–Raphson method was used in the preceding examples, other approximation techniques could have been used equally as well. Most of these techniques, however, require that the iteration be initiated in the neighborhood of the solution to assure convergence. In those instances where many coefficients are desired, difficulty may be experienced in achieving convergence even with initial values very close to the true solutions.

The dimensionless parameter σ appearing in the thermal diffusivity–temperature relationship (2.5) is the product of the reference temperature T_s and the slope of the dimensional diffusivity–temperature curve. Values of σ used in the examples were chosen for illustrative purposes only and not to represent the properties of a particular solid material. These values are, however, valid for many metals and structural materials when the reference temperature T_s is of the order of several hundred to a thousand degrees.

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Résumé—Une formulation variationnelle est obtenue et appliquée à des problèmes de conduction de chaleur à travers des solides. Cette formulation a, comme équation d'Euler–Lagrange, l'équation générale de la chaleur où la conductibilité thermique, la masse volumique et la chaleur spécifique peuvent dépendre de la température. Le principe variationnel peut être appliqué à des solutions en régime permanent ou dépendant du temps et il est valable pour des conditions aux limites dans lesquelles la température est spécifiée, soit constante, soit dépendant du temps, ou lorsque le flux de chaleur est nul. La distribution de température macroscopique est déterminée pour deux problèmes en employant cette technique. Le premier de ces problèmes est celui d'une plaque rectangulaire finie dont la température initiale est constante et dont la température des bords est portée à zéro pour des temps positifs. Dans le second exemple, la formulation variationnelle est employée pour obtenir une distribution de température en régime permanent pour une plaque rectangulaire finie dont les côtés sont maintenus à des températures constantes mais différentes. Dans ces deux applications, on a supposé que la conductivité thermique varie linéairement avec la température et l'on a supposé que la masse volumique et la chaleur spécifique avaient des valeurs uniformes. La comparaison des résultats obtenus en employant la formulation variationnelle avec ceux obtenus par des techniques de différences finies montre une corrélation excellente dans une certaine gamme de variation du coefficient de température de la conductivité thermique.

Zusammenfassung—Mit Hilfe der Variationsrechnung wird ein Formalismus abgeleitet und auf Probleme der Wärmeleitung in Festkörpern angewandt. Die Formulierung enthält als Euler–Lagrangesche Differentialgleichung die allgemeine Wärmeleitungsgleichung, wobei die Wärmeleitfähigkeit, die Dichte und die spezifische Wärme temperaturabhängig sein können. Das Variationsprinzip kann zur Gewinnung stationärer, wie instationärer Lösungen dienen; est gilt für Randbedingungen, bei denen die Temperatur entweder als fester oder als zeitabhängiger Wert vorgeschrieben ist oder der Wärmestrom Null ist.

Für zwei Probleme wird die makroskopische Temperaturverteilung unter Verwendung dieser Methode ermittelt. Im ersten Fall ist eine endliche rechteckige Platte vorgegeben; die Anfangstemperatur der Platte sei konstant und für Zeiten grösser Null werden die Ränder auf der Temperatur Null gehalten. Im zweiten Beispiel wird die Formulierung als Variationsproblem dazu benutzt, die stationäre Temperaturverteilung in einer endlichen rechtwinkligen Platte zu gewinnen, wenn die Seitenflächen auf festen aber unterschiedlichen Temperaturen gehalten werden. In beiden Anwendungsfällen wird angenommen, dass die Wärmeleitfähigkeit linear mit der Temperatur veränderlich ist und die Dichte und die spezifische Wärme feste Werte haben.

Ein Vergleich der nach dem Variationsprinzip gewonnenen Ergebnisse mit solchen aus einem Differenzenverfahren zeigt für einen gewissen Bereich des Temperatur-Wärmeleitfähigkeitskoeffizienten ausgezeichnete Übereinstimmung.

Аннотация—Полученный варицонный метод применяется к задачам теплопроводности твердых тел. Этот метод, основанный на уравнении Эйлера–Лагранжа, содержит общее уравнение переноса тепла, в котором теплопроводность, плотность и удельная теплоемкость могут зависеть от температуры. Вариационный принцип применяется для решения как стационарных так и нестационарных задач и справедлив для граничных условий, в которых задается температура, причем она постоянна или зависит от времени, а также для случая, когда тепловой поток равен нулю. С помощью данного метода находится распределение температуры для следующих случаев: (1) ограниченной пластины с постоянной начальной температурой, температура боковых поверхностей равна нулю когда время больше нуля; (2) получено стационарное температурное распределение для ограниченной пластины, когда боковые поверхности поддерживаются при фиксированных и различных температурах. В обоих этих случаях предполагается, что коэффициент теплопроводности линейно изменяется с температурой, а плотность и удельная теплоемкость-величины постоянные. Сравнение полученных вариационным методом результатов со значениями, полученными методом конечных разностей, показывает хорошее согласование для коэффициента температуропроводности.